MATHEMATICS

ON THE SIMULTANEOUS REPRESENTATION OF A GIVEN PAIR OF INTEGERS AS THE SUM RESPECTIVELY OF FOUR INTEGERS AND THEIR SQUARES. I.

BY

LILY SESHU

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Introduction

In this paper an expression is obtained for the arithmetic function $r_s(n, m)$ which represents the number of solutions of the diophantine equations:

(a) $$x_1^2 + x_2^2 + \dots + x_s^2 = n,$$

(b) $$x_1 + x_2 + \dots + x_s = m,$$

when the value of s is four.

KLOOSTERMAN used the circle method of Hardy and Littlewood to obtain heuristically a function $\varrho_s(n, m)$ which approximates $r_s(n, m)$. When $3 \leqslant s \leqslant 8$, it turns out that $\varrho_s(n, m) = r_s(n, m)$. KLOOSTERMAN himself [1] showed this for $s = 3, 5, 7$ and BRONKHORST [1] did the same for $s = 6, 8$. Van der Blij used an algebraic method to obtain an exact formula for $r_4(n, m)$. LOMADZE [1] on the other hand summed the singular series to obtain a closed expression for $\varrho_s(n, m)$ for $s \geqslant 3$. His results together with those of Van der Blij indicate that $r_4(n, m) = \varrho_4(n. m)$.

In this paper the same result is proved directly, utilizing the technique Bronkhorst and Kloosterman (loc. cit.). A convergence problem arises in the process similar to that encountered by several other authors in the solution of problems of a similar nature. See for instance KLOOSTERMAN (loc. cit.), H. STREEFKERK [1], P. T. BATEMAN [1], and some others. The technique used to overcome the difficulty, namely the introduction of a convergence-producing factor into the terms of a formal series, goes back to HECKE [1].

Some symbols used

Let a, b, q, x be integers.

$$e\left(\frac{a}{q}\right) \ : \ e^{2\pi i a/q}.$$

$$\sum_{a(q)} f(a) \text{ or } \sum_{a(\mathrm{mod}\, q)} f(a \ : \sum_{a=0}^{q-1} f(a).$$

$$\sum_{a(q)}' f(a) \text{ or } \sum_{a \bmod q}' f(a) \ : \sum_{\substack{a=0 \\ (a,q)=1}}^{q-1} f(a).$$

If $a \neq 0$,

$\left(\frac{a}{b}\right)$: Jacobi or Kronecker symbol as appropriate, $(b>1)$.

$$\left(\frac{a}{1}\right) = 1.$$

$$\sum_m f(m) \ : \sum_{m=-\infty}^{\infty} f(m).$$

$$S(a, q, b) \ : \sum_{x \bmod q} e\left(\frac{ax^2+bx}{q}\right).$$

Section 1. *Definitions and basic properties.*

In this section the functions $\theta_z(v|\tau)$ and $\theta(v|\tau)$ are defined. $\theta_z(v|\tau)$ is defined by means of a double Laurent series in $e^{2\pi iv}$ and $e^{\pi i\tau}$ where v, τ and z are complex variables with $\mathscr{I}(\tau)>0$ – the coefficient of $e^{2\pi ivm}\, e^{\pi i\tau n}$ being $\varrho_4(n, m)$. In the initial definition of $\theta_z(v|\tau)$, z is restricted to the half plane $\mathscr{R}(z)>{}^1/_2$. Using a certain transformation on $\theta_z(v|\tau)$, it is shown that the latter may be continued analytically into the region $\mathscr{R}(z)>-{}^1/_2$ and that $\theta(v|\tau)$ is identical with the value of $\theta_z(v|\tau)$ at $z=0$.

Since the lines of work are necessarily similar to those of Bronkhorst [1] and Kloosterman [1], every attempt has been made to retain the same notation. For the same reason the subscript s is used rather than four, even though in the whole of this paper s is assumed to be four unless specifically otherwise stated.

We need certain results concerning $\varrho_s(n, m)$ and the singular series prior to defining $\theta(v|\tau)$.

From the work of Kloosterman [1, p. 325, p. 333] we obtain an expression for $\varrho_s(n, m)$. Let

$$\Delta = ns - m^2.$$

If $\Delta \leqslant 0$, we put:

$$r_s(n, m) = \varrho_s(n, m) = \begin{cases} 0 \text{ for } \Delta<0 \text{ or } s \nmid m \\ 1 \text{ for } \Delta=0 \text{ and } s|m. \end{cases} \tag{1}$$

If $\Delta>0$, we put

$$\varrho_s(n, m) = \frac{\pi^{(s-1)/2}\Delta^{(s-3)/2}}{s^{(s-2)/2}\,\Gamma\left(\frac{s-1}{2}\right)} \sum_{q=1}^{\infty} \sum_{a(q)}' \sum_{b(q)} \left(\frac{S(a, q, b)}{q}\right)^s e\left(\frac{-na-mb}{q}\right). \tag{2}$$

In future we denote by η_s the factor

$$\frac{\pi^{(s-1)/2}}{s^{(s-2)/2}\,\Gamma\left(\frac{s-1}{2}\right)}.$$

Then $\varrho_s(n, m)$ is related to the singular series $\mathfrak{S}_s(n, m)$ thus:

$$\varrho_s(n, m) = \eta_s \, \mathfrak{S}_s(n, m) \cdot \Delta^{(s-3)/2}. \tag{3}$$

If we define:

$$B_q(n, m) = \sum_{a(q)}' \sum_{b(q)} \left(\frac{S(a, q, b)}{q}\right)^s e\left(\frac{-na-mb}{q}\right), \tag{4}$$

then we have:

$$\varrho_s(n, m) = \eta_s \sum_{q=1}^{\infty} B_q(n, m) \cdot \Delta^{(s-3)/2}.$$

Kloosterman shows [1, p. 323 and p. 333] that

$$B_{q_1q_2}(n, m) = B_{q_1}(n, m) \cdot B_{q_2}(n, m), \tag{5}$$

provided $(q_1, q_2) = 1$.

G. A. Lomadze [1] considers the convergence and effects the summation of the singular series for $s \geqslant 3$. $\mathfrak{S}_s(n, m)$ is absolutely convergent only for $s \geqslant 5$ but converges for $s = 3, 4$. Lomadze's results for $s = 4$ are stated in equations (7a) to (9) below. These equations yield the following observations: If q is odd, $B_q(n, m)$ depends only on $\Delta = ns - m^2$. If q is even but $2^4 \nmid q$, $B_q(n, m)$ depends on the residue classes modulo 4 (in some cases, modulo 8) to which n and m belong. If $2^4 | q$, $B_q(n, m)$ depends on Δ and the residue classes of m modulo 4.

In what follows the argument of B_q may be dropped if no ambiguity results.

If h is a prime, we define:

$$\chi_h(n, m) = 1 + B_h + B_{h^2} + \dots \tag{6}$$

For the rest of this paper p denotes an odd prime.

Lomadze's results for $B_q(n, m)$ when $s = 4$:

$$q = 2^\lambda, \quad 1 \leqslant \lambda \leqslant 3. \tag{i}$$

$$B_2 = (-1)^{n+m} \tag{7a}$$

$$B_{2^2} = \begin{cases} 0 & \text{if } 2 \nmid m, \text{ or } 2 | m \text{ and } 2 \nmid n \\ 1 & \text{if } 2 | m, \ 2 \| n \\ -1 & \text{if } 2 | m, \ 4 | n. \end{cases} \tag{7b}$$

$$B_{2^3} = \begin{cases} 0 & \text{if } 2 \nmid m, \text{ or } 4 | m, \text{ or } 2 \| m \text{ and } 4 \nmid n \\ 1 & \text{if } 2 \| m, \ 4 \| n \\ -1 & \text{if } 2 \| m, \ 8 | n. \end{cases} \tag{7c}$$

$$q = 2^\lambda, \quad \lambda > 3. \tag{ii}$$

$$(7d)\qquad B_{2^\lambda} = \begin{cases} 0 & \text{if } 2 \nmid m, \text{ or } 2 \| m, \\ & \text{or } 4 \mid m,\ 2 \nmid \lambda,\ 2^{\lambda-1} \nmid \Delta, \\ & \text{or } 4 \mid m,\ 2 \mid \lambda,\ 2^\lambda \nmid \Delta. \\ \dfrac{-C\cdot 2}{2^{(\lambda-1)/2}} & \text{if } 4 \mid m,\ 2 \nmid \lambda,\ 2^{\lambda-1} \mid \Delta; \\ \dfrac{-A\cdot 2}{2^{\lambda/2}} & \text{if } 4 \mid m,\ 2 \mid \lambda,\ 2^\lambda \mid \Delta. \end{cases}$$

where

$$A = \begin{cases} 1 \text{ if } \Delta/2^\lambda \equiv 0,3 \pmod 4) \\ -1 \text{ if } \Delta/2^\lambda \equiv 1, 2 \pmod 4). \end{cases}$$

$$C = \begin{cases} -1 \text{ if } \Delta/2^{\lambda-1} \equiv 3 \pmod 8) \\ 1 \text{ if } \Delta/2^{\lambda-1} \equiv 7 \pmod 8) \\ 0 \text{ all other cases.} \end{cases}$$

$$q = p^\lambda.$$

$$(7e)\quad B_{p^\lambda} = \begin{cases} 0 & \text{if } 2 \mid \lambda,\ p^{\lambda-1} \nmid \Delta, \\ & \text{or if } 2 \nmid \lambda \text{ and either } p^{\lambda-1} \nmid \Delta \text{ or } p^\lambda \mid \Delta. \\ \dfrac{-p^{(\lambda-2)/2}}{p^\lambda} & \text{if } 2 \mid \lambda,\ p^{\lambda-1} \| \Delta \\ \dfrac{p^{(\lambda-2)/2}(p-1)}{p^\lambda} & \text{if } 2 \mid \lambda,\ p^\lambda \mid \Delta \\ \left(\dfrac{-p^{-\lambda+1}\Delta}{p}\right) \dfrac{p^{(\lambda-1)/2}}{p^\lambda} & \text{if } 2 \nmid \lambda,\ p^{\lambda-1} \| \Delta. \end{cases}$$

Equations (7a) to (7d) give the following results:
If $2^\alpha \| \Delta$, $2^{-\alpha}\Delta = \Delta_1$, and $\alpha \geqslant 0$ is an integer, then

$$(8)\qquad \chi_2(n, m) = \begin{cases} 0 & \text{if } m \not\equiv n \pmod 2) \\ 2 & \text{if } m \equiv n \equiv 1 \pmod 2) \\ 3 & \text{if } 2 \mid m,\ 2 \| n. \\ 1-(-1)^{n/4} & \text{if } 2 \| m,\ 4 \mid n, \\ 3.2^{(3-\alpha)/2} & \text{if } m \equiv n \equiv 0 \pmod 4),\ \alpha \text{ is odd and } > 4 \\ E.\ 2^{-\alpha/2} & \text{if } m \equiv n \equiv 0 \pmod 4),\ \alpha \text{ is even and } \geqslant 4. \end{cases}$$

where

$$E = \begin{cases} 0 & \text{if } \Delta_1 \equiv 7 \pmod 8) \\ 4 & \text{if } \Delta_1 \equiv 3 \pmod 8) \\ 6 & \text{if } \Delta_1 \equiv 1,5 \pmod 8). \end{cases}$$

For each odd prime p, let β_p be the non-negative integer for which $p^{\beta_p} \| \Delta$ and let

$$\Delta_2 = \prod_{\substack{p \mid \Delta \\ \beta_p \equiv 1 \pmod 2)}} p^{\beta_p}, \quad \Delta_3 = \prod_{\substack{p \\ \beta_p \equiv 0 \pmod 2)}} p^{\beta_p}, \quad D = \prod_{p \mid \Delta_2} p.$$

Then Lomadze's results give:

$$(9)\quad \begin{cases} \varrho_4(n,m) = \dfrac{4}{\pi}\,\Delta^{1/2}\,\chi_2 \displaystyle\prod_{p\mid\Delta_2} \frac{1-p^{-(\beta_p+1)/2}}{1-p^{-1}} \\ \displaystyle\prod_{p\mid\Delta_3}\left[\frac{1-p^{-\beta_p/2}}{1-p^{-1}}\left\{1-\left(\frac{-2^\alpha D}{p}\right)p^{-1}\right\}+p^{-\beta_p/2}\right] \sum_{\substack{q=1\\(q,4D)=1}}^{\infty} \frac{(-2^\alpha D/q)}{p}. \end{cases}$$

That the infinite series in (9) is convergent and replaceable by a finite sum is shown in lemma 1 below.

Lemma 1. If D is a positive, odd, square-free integer and α a non-negative integer, then:

$$(10)\qquad \sum_{\substack{q=1\\(q,4D)=1}}^{\infty}\left(\frac{-2^\alpha D}{q}\right)\frac{1}{q} = \begin{cases} \dfrac{-\pi}{(8D)^{3/2}}\displaystyle\sum_{r=1}^{8D-1}\left(\frac{-8D}{r}\right)r \text{ if } 2\nmid\alpha \\ \left[1-\left(\dfrac{-D}{2}\right)\dfrac{1}{2}\right]\dfrac{-\pi}{D^{3/2}}\displaystyle\sum_{r=1}^{D-1}\left(\frac{-D}{r}\right)r \text{ if} \\ \quad 2\mid\alpha \text{ and } D\equiv 3 \pmod 4 \\ \dfrac{-\pi}{(4D)^{3/2}}\displaystyle\sum_{r=1}^{4D-1}\left(\frac{-4D}{r}\right)r \text{ if } 2\mid\alpha \text{ and} \\ \quad D\equiv 1 \pmod 4). \end{cases}$$

Proof. If d is any integer such that $d\equiv 0$ or $1 \pmod 4$ and not a square, we may define (cf. LANDAU [2, p. 145])

$$K(d) = \sum_{n=1}^{\infty}\left(\frac{d}{n}\right)\frac{1}{n}.$$

If d is further a fundamental discriminant number and $d<0$, we have:

$$K(d) = \frac{-\pi}{|d|^{3/2}}\sum_{r=1}^{|d|-1}\left(\frac{d}{r}\right)r.$$

(cf. LANDAU [2, p. 172 and p. 177]).

Now, D being odd, $-8D$ is always a fundamental discriminant number; if $D\equiv 1 \pmod 4$, $-4D$ is a fundamental discriminant number; and if $D\equiv 3 \pmod 4$ then $-D$ is a fundamental discriminant number and:

$$K(-D)\,[(1-(-D/2)^1/_2] = K(-4D),$$

by Satz 214 of LANDAU [2, p. 172].

The proof of equation (10) follows.

From (8) and (9) we get an estimate for $\varrho_4(n,m)$.

$|\varkappa_2(n,m)| \leqslant 3,$

$$\prod_{p\mid\Delta_2}\frac{1-p^{-(\beta_p+1)/2}}{1-p^{-1}} < \prod_{p\mid\Delta_2}\frac{1}{1-p^{-1}} \leqslant \max\,[1, N\log|\Delta_2|],$$

where N is a positive numerical constant (cf. HARDY and WRIGHT [1, § 22.9]).

Similarly

$$\prod_{p|\Delta_3}\left[\frac{1-p^{-\beta_p/2}}{1-p^{-1}}\left(1-\left(\frac{-2^\alpha D}{p}\right)\frac{1}{p}\right)+p^{-\beta_p/2}\right]\leqslant \max\,[1,\, N^2\log^2|\Delta_3|]$$

and

$$\left|\sum_{q=1}^{\infty}\left(\frac{-2^\alpha D}{q}\right)\frac{1}{q}\right|<\log|8D|$$

(cf. BATEMAN [1, p. 85]).

From the above we have ($\varepsilon>0$)

(11) $$\varrho_s(n, m) = O(\Delta^{1/2+\varepsilon}).$$

We now define $\theta(v|\tau)$ thus:

(12) $$\theta(v|\tau)=\sum_{\substack{n\\ m^2\leqslant ns}}\sum_{m}\varrho_s(n,m)\,w^n\,e^{2\pi ivm},$$

where in the above equation, as in the rest of this paper, we assume:

$$\begin{aligned} w &= e^{\pi i\tau}\\ \tau &= \alpha+i\beta\\ v &= v_1+iv_2,\end{aligned}$$

with α, β, v_1 and v_2 real and $\beta>0$.

We decompose $\theta(v|\tau)$ into the auxiliary functions $\psi(v|\tau)$ and $\vartheta/(v|\tau)$ as follows:

(13) $$\theta(v|\tau)=\vartheta_0(v|\tau)+\psi(v|\tau)$$

where

(14) $$\vartheta_0(v|\tau)=\sum_{\substack{m\\ m\equiv 0(s)}} w^{m^2/s}\,e^{2\pi ivm}$$

(15) $$\psi(v|\tau)=\sum_{\substack{n\\ m^2<ns}}\sum_{m}\varrho_s(n,m)\,w^n\,e^{2\pi ivm}.$$

(13) follows readily from (12), (14) and (15) when we recall (1). $\vartheta_0(v|\tau)$ is clearly analytic and regular in v and τ ($\mathscr{I}(\tau)>0$). Using (11) we see that the right hand side of (15) is majorized by

$$C_0(\varepsilon)\sum_{n=1}^{\infty}e^{-\pi n\beta}\cdot n^{1-\varepsilon}\cdot e^{2\pi|v_2|\sqrt{ns}}\qquad(\varepsilon>0,\ \beta>0),$$

where $C_0(\varepsilon)$ is real, positive and a function of ε alone. Hence $\psi(v|\tau)$ and therefore $\theta(v|\tau)$ are analytic functions of v and τ, regular for $\mathscr{I}(\tau)>0$.

It is convenient to replace the summation variable n by $k=ns—m^2$

in equation (15). Doing so and substituting for $\varrho_s\left(\frac{k+m^2}{s}, m\right)$ from (2) we obtain:

$$(16)\qquad \begin{cases} \psi(v|\tau) = \eta_s \sum\limits_m w^{m^2/s} e^{2\pi i v m} \sum\limits_{\substack{k=1\\ k\equiv -m^2(s)}} k^{(s-3)/2} w^{k/s} \\ \qquad \sum\limits_{q=1}^{\infty} \sum\limits_{a(q)}' e\left(\frac{-ka}{qs}\right) \sum\limits_{b(q)} \left(\frac{S(a, q, b)}{q}\right)^s e\left(\frac{-m^2a-mbs}{qs}\right). \end{cases}$$

This may be re-written as:

$$(17)\qquad \psi(v|\tau) = \sum_m w^{m^2/s} e^{2\pi i v m} \psi_m(\tau)$$

where

$$(18)\qquad \psi_m(\tau) = \eta_s \sum_{\substack{k=1\\ k\equiv -m^2(s)}}^{\infty} k^{(s-3)/2} w^{k/s} \sum_{q=1}^{\infty} \sum_{a(q)}' e\left(\frac{-ka}{qs}\right) \sum_{b(q)} \left(\frac{S(a, q, b)}{q}\right)^s e\left(\frac{-m^2a-mbs}{qs}\right).$$

Now let

$$(19)\qquad \sigma(a, q, m) = q^{(s-1)/2} \sum_{b(q)} \left(\frac{S(a, q, b)}{q}\right)^s e\left(\frac{-m^2a-mbs}{qs}\right).$$

This definition of $\sigma(a, q, m)$ is formally identical with that of Bronkhorst who has developed its properties [1, pp. 11−13] and his work is easily seen to be valid when $s=4$.

We use these results beginning with the fact that $\sigma(a, q, m)$ has period s in m. That $\psi_m(\tau)$ also has period s in m follows when it is written in the form:

$$(20)\qquad \psi_m(\tau) = \eta_s \sum_{\substack{k=1\\ k\equiv -m^2(s)}}^{\infty} k^{(s-3)/2} w^{k/s} \sum_{q=1}^{\infty} q^{(1-s)/2} \sum_{a(q)}' e\left(\frac{-ka}{qs}\right) \sigma(a, q, m).$$

Hence $\psi(v|\tau)$ can be written as:

$$(21)\qquad \begin{cases} \psi(v|\tau) = \sum\limits_{\mu(s)} \psi_\mu(\tau) \sum\limits_{m\equiv\mu(s)} w^{m^2/s} e^{2\pi i v m} \\ \qquad\quad = \sum\limits_{\mu(s)} \psi_\mu(\tau)\, \vartheta_\mu(v|\tau) \end{cases}$$

where (consistent with (14))

$$(22)\qquad \vartheta_\mu(v|\tau) = \sum_{m\equiv\mu(s)} w^{m^2/s} e^{2\pi i v m}.$$

From the above we obtain easily

$$(23)\qquad \theta(v|\tau) = \vartheta_0(v|\tau) + \sum_{\mu(s)} \psi_\mu(\tau) \cdot \vartheta_\mu(v|\tau).$$

To investigate the series in (18) we write it as: (cf. (4))

$$(24)\qquad \psi_m(\tau) = \eta_s \sum_{\substack{k=1\\ k\equiv -m^2(s)}}^{\infty} w^{k/s} k^{(s-3)/2} \sum_{q=1}^{\infty} B_q\left(\frac{k+m^2}{s}, m\right).$$

Since, when $s=4$ and $p \nmid k$

$$B_p = \left(\frac{-k}{p}\right)\frac{1}{p}.$$

The inner sum in (24) is not absolutely convergent.

For $s>5$ however the double series in (24) can be shown to be absolutely convergent and in this case one can reverse the order of summation in (24) obtaining:

$$(25)\qquad \psi_m(\tau) = \eta_s \sum_{q=1}^{\infty} \sum_{a(q)}' q^{(1-s)/2} \cdot \sigma(a, q, m) \sum_{\substack{k=1 \\ k \equiv -m^2(s)}}^{\infty} e\left(\frac{-ka}{qs}\right) k^{(s-3)/2} w^{k/s} \qquad (s>5).$$

Using a formula of LIPSCHITZ [1] on the series in k, BRONKHORST [1, pp. 9–10] shows that:

$$(26)\qquad \begin{cases} \psi_m(\tau) = \sum_{q=1}^{\infty} \sum_{a(q)}' q^{(1-s)/2}\, \sigma(a, q, m)\, s^{-1/2}\, e\left(\frac{s-1}{8}\right) 2^{-(s-1)/2} \\ \qquad\qquad \sum_r \frac{e(-m^2 r/s)}{(\tau/2-a/q-r)^{(s-1)/2}}, \end{cases}$$

where

$$0 < \arg \sqrt{\tau/2 - \mathrm{a}/q - r} < \pi/2.$$

While the above transformations have not been justified for $s=4$, they motivate our definition of the function $\psi_{m,z}(\tau)$ below.

Let $z=\sigma+it$ and put:

$$(27)\qquad \begin{cases} \psi_{m,z}(\tau) = \dfrac{s^{-1/2}\, e\left(\frac{s-1}{8}\right)}{2^{(s-1)/2}} \sum_{q=1}^{\infty} \dfrac{2^{\delta(q)z}}{q^{(s-1)/2+z}} \sum_{a(q)}' \sigma(a, q, m) \\ \qquad\qquad \sum_r \dfrac{e(-m^2 r/s)}{\left(\frac{\tau}{2}-\frac{a}{q}-r\right)^{(s-1)/2} 2^z \left|\frac{\tau}{2}-\frac{a}{q}-r\right|^z} \qquad (s=4,\ \sigma>1/2). \end{cases}$$

It will now be shown that $\psi_{m,z}(\tau)$ is analytic in z for $\mathscr{R}(z)>1/2$ (v and τ fixed).

Making the change of variable $a'=a+rq$ and using the periodocity of $S(a, q, b)$ in a it is easily shown (cf. BRONKHORST [1, p. 10]) that: [1])

$$(28)\qquad \psi_{m,z}(\tau) = s^{-1/2}\, e\left(\frac{s-1}{8}\right) \sum_{\substack{q=1 \\ (a,q)=1}}^{\infty} \sum_a \frac{2^{\delta(q)z}\, \sigma(a, q, m)}{(q\tau-2a)^{(s-1)/2}\, |q\tau-2a|^z}.$$

We need estimates for $\sigma(a, q, m)$. From Bronkhorst's results (pp. 11–13) we have:

$$(29)\qquad |\sigma(a, q, m)| = \begin{cases} q^{-1/2} |S(-as, q, -2am)|, \text{ if } q \equiv 1 \pmod 2 \\ (q/2)^{-1/2}\, 2^{(s-1)/2} |S(-2as, q/2, -2am)|, \text{ if } q=2 \pmod 4 \\ (q)^{-1/2} |S(-as, q, -2am)|\, 2^{(s-2)/2}, \text{ if } q=0 \pmod 4). \end{cases}$$

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[1]) The inner sum in (28) is absolutely convergent for $\mathscr{R}(z)$ sufficiently large.

By well-known bounds for $|S(a, q, b)|$ it now follows that:

$$|\sigma(a, q, m)| \leqslant 2^{(s-1)/2}. \tag{30}$$

Thus the series defining $\psi_{m,z}(\tau)$ in (28) is majorized by:

$$2^{(s-1)/2} s^{-1/2} \sum_{\substack{q=1 \\ (a,q)=1}}^{\infty} \sum_{a} \frac{2^{\delta(q)\sigma}}{|q\tau-2a|^{(s-1)/2+\sigma}}. \tag{31}$$

which converges for $\frac{s-1}{2}+\sigma>2$ (cf. L. R. FORD [1, pp. 151–52]). Hence when $s=4$, the series converges uniformly for $\sigma>1/2$, so that $\psi_{m,z}(\tau)$ is a regular, analytic function of z in the region: $\mathscr{R}(z)>1/2$.

Since $\psi_{m,z}(\tau)$, like $\psi_m(\tau)$, has periodicity s in m, we define:

$$\psi_z(v|\tau) = \sum_{\mu \bmod s} \psi_{\mu,z}(\tau) \cdot \vartheta_\mu^{*}(v|\tau) \qquad (\mathscr{R}(z)>1/2), \tag{32}$$

$$\theta_z(v|\tau) = \vartheta_0(v|\tau)+\psi_z(v|\tau), \qquad (\mathscr{R}(z)>1/2). \tag{33}$$

We proceed to show that $\psi_{m,z}(\tau)$ (and hence also $\psi_z(v|\tau)$ and $\theta_z(v|\tau)$ can be continued analytically into the region $\mathscr{R}(z)>-1/2$ and that:

$$\psi_{m,z}(\tau)|_{z=0} = \psi_{m,0}(\tau) = \psi_m(\tau).$$

In equation (27) we consider the inner sum and define:

$$P' = \sum_{r} \frac{e(-m^2 r/s)}{(\tau/2-a/q-r)^{(s-1)/2}\,|\tau/2-a/q-r|^z}. \tag{34}$$

Thus

$$\psi_{m,z}(\tau) = \frac{s^{-1/2}\, e\left(\frac{s-1}{8}\right)}{2^{(s-1)/2+z}} \sum_{q=1}^{\infty} \sum_{a(q)}{}' \frac{\sigma(a, q, m)\, 2^{\delta(q)z}}{q^{(s-1)/2+z}} P' \qquad (\mathscr{R}(z)>1/2). \tag{35}$$

Now, (34) is unaltered when $-r$ is replaced by r. We do this and apply to (34) the Poisson summation formula (L. J. MORDELL [2]) to obtain:

$$P' = \sum_{n} \int_{-\infty}^{\infty} \frac{e\left(\frac{m^2 x}{s}\right) e(-nx)\, dx}{(\tau/2-a/q+x)^{(s-1)/2}\,|\tau/2-a/q+x|^z}.$$

On writing:

$$k = ns - m^2, \quad u = x - a/q\ ,$$

We have:

$$\left\{\begin{aligned} P' &= \sum_{k \equiv -m^2 (s)} \int_{-\infty}^{\infty} \frac{e(-ku/s) \cdot e(-ka/qs)\, du}{(\tau/2+u)^{(s-1)/2}\,|\tau/2+u|^z} \\ &= \sum_{k\equiv -m^2(s)} e(-ka/qs) \cdot I(k, \tau, z), \end{aligned}\right. \tag{36}$$

where (36) defines I (k, τ, z) in an obvious way.

In what follows S will denote an arbitrary but fixed compact subset of the region $\mathscr{R}(z) > -1/2$, containing the set $\{z\colon -1/4 < \sigma < 1,\ |t| < 1\}$. The constants $C_1, C_2, \ldots C_i \ldots$ are assumed to be positive and depend at most on S and τ both of which are assumed to be fixed for the rest of this section, $\mathscr{I}(\tau)$ being positive.

Lemma 2. The function $I(k, \tau, z)$ has the following properties:

a) For each k and τ ($\mathscr{I}(\tau) > 0$), $I(k, \tau, z)$ is an analytic function of z, regular for $\mathscr{R}(z) > -1/2$.

b) In the set S,

$$|I(k, \tau, z)| < C_1\, e^{-C_2|k|}.$$

Lemma 3.

$$I(k, \tau, 0) = \begin{cases} 0 \text{ if } k \leqslant 0, \\ \dfrac{e\left(\dfrac{1-s}{8}\right)(2\pi)^{(s-1)/2}\,(k/s)^{(s-3)/2}\,e^{\pi i \tau k}}{\Gamma\left(\dfrac{s-1}{2}\right)} \\ \text{if } k > 0. \end{cases} \tag{37}$$

Lemmas 2 and 3 appear in almost the present form in the thesis of H. Streefkerk [1, p. 31]. The method of proof was indicated by Hecke [1, pp. (224–226)]. See also part I of same paper (p. 111). A detailed proof can be found in Seshu [1]), [1, p. 24].

From (35) and (36) we get:

$$\begin{cases} \psi_{m,z}(\tau) = \dfrac{s^{-1/2}\, e\left(\dfrac{s-1}{8}\right)}{2^{(s-1)/2+z}} \displaystyle\sum_{q=1}^{\infty} \sum_{a(q)}{}' \frac{\sigma(a, q, m)\, 2^{\delta(q)z}}{q^{(s-1)/2+z}} \\ \qquad \displaystyle\sum_{k \equiv -m^2(s)} I(k, \tau, z)\, e\left(\frac{-ka}{qs}\right), \qquad (\sigma > 1/2). \end{cases} \tag{38}$$

From the estimates for $I(k, \tau, z)$ in lemma 2b), the double series in (38) is absolutely convergent for $\mathscr{R}(z) > 1/2$. On interchanging summations and using (19) and (4) we get $\left(\text{taking, as usual, } \dfrac{k+m^2}{s} = n\right)$ that:

$$\psi_{m,z}(\tau) = \frac{s^{-1/2}\, e\left(\dfrac{s-1}{8}\right)}{2^{(s-1)/2+z}} \sum_{k \equiv -m^2(s)} I(k, \tau, z) \sum_{q-1}^{\infty} \frac{B_q(n, m)\, 2^{\delta(q)z}}{q^z}. \tag{39}$$

Define

$$G_z(n, m) = \sum_{q=1}^{\infty} \frac{B_q(n, m)\, 2^{\delta(q)z}}{q^z} \qquad (\mathscr{R}(z) > 1/2). \tag{40}$$

From (7a) through (7e), it follows that:

$$|B_q(n, m)| < 4q^{-1/2}.$$

where

$$q = p^\lambda \text{ or } q = 2^\lambda,\ (\lambda, \text{ a positive integer}).$$

[1]) See also Bateman [1].

Consequently one may use the multiplicativity of $B_q(n, m)$ to write:

$$G_z(n, m) = \chi_{2,z}(n, m) \cdot F_z(n, m) \qquad (\mathscr{R}(z) > 1/2), \tag{41}$$

where

$$\chi_{2,z}(n, m) = 1 + 2^z \sum_{\lambda=1}^{\infty} B_{2^\lambda}\, 2^{-\lambda z}. \tag{42}$$

We now show that $G_z(n, m)$ is an analytic function, regular for $\mathscr{R}(z) > -1/2$ and under a bound, depending at most on k, in the set S, where, as before, $k = ns - m^2$.

Lemma 3. $F_z(n, m)$ is an analytic function of z which is:

a) regular for $\mathscr{R}(z) > -1/2$ when $-k$ is not a square,

b) regular in the half-plane $\mathscr{R}(z) > -1/2$ except at the point $z = 0$, when $k = 0$ or when $k < 0$ and $-k$ is a square. In these cases $F_z(n, m)$ has a simple pole at $z = 0$.

Proof. For convenience we define:

$$\chi_{p,z}(n, m) = 1 + \sum_{\lambda=1}^{\infty} B_{p^\lambda}(n, m)\, p^{-\lambda z} \qquad (\sigma > 1/2). \tag{43}$$

By (7e) $\chi_{p,z}(n, m)$ depends only on k and not on n and m separately[1].

Inspection of (7e) shows that when $k \neq 0$, the right hand side of (43) has at most $\beta + 2$ non-zero terms if $p^\beta \| k$ (β being a non-negative integer). Thus $\prod_{p|k} \chi_{p,z}(k)$ is an entire function of z when $k \neq 0$.

Since

$$|B_{p^\lambda}(k)| < p^{-\lambda/2},$$

an easy calculation shows that:

$$|\chi_{p,z}(k)| < C_3 p^\beta \qquad (\mathscr{R}(z) > -1/2). \tag{44}$$

When $k \neq 0$, (7e) gives:

$$\prod_{p \nmid k} \chi_{p,z}(k) = \prod_{p \nmid k} \left(1 + \left(\frac{-k}{p}\right) p^{-1-z}\right) \qquad (\mathscr{R}(z) > 0). \tag{45}$$

By the multiplicativity of $B_{q,z}(n, m)$ one gets:

$$\tag{46} \left\{ \begin{aligned} F_z(n, m) &= \sum_{\substack{q=1 \\ q \equiv 1(2)}}^{\infty} B_q(n, m)\, q^{-z} = \prod_{p|k} \chi_{p,z} \cdot \prod_{p \nmid k} \chi_{p,z} \\ &= \prod_{p|k} \chi_{p,z} \cdot \prod_{p \nmid k} \left(1 + \left(\frac{-k}{p}\right) p^{-1-z}\right) \\ &= \prod_{p|k} \chi_{p,z} \cdot \sum_{q=1}^{\infty} \left(\frac{-4k}{q}\right) q^{-1-z} \mu^2(q) \qquad (\mathscr{R}(z) > 0), \end{aligned} \right.$$

[1]) In the rest of the section, we write $B_{p^\lambda}(k)$ ($\chi_{p,z}(k)$) in place of $B_{p^\lambda}(n, m)$ ($\chi_{p,z}(n, m)$), whenever convenient.

where μ is the Möbius function. Now $\left(\frac{-4k}{q}\right)$ is a residue class character modulo $|4k|$ and is the principal character only if $-4k$ is a square, (LANDAU [2, satz. 99]). We denote this character (defined on the positive integers) by $\chi(q)$. STREEFKERK [1, hulpt. 24] has shown that:

$$(47)\qquad \begin{cases} \sum_{q=1}^{\infty} \chi(q)\,\mu^2(q)\,q^{-1-z} = L(1+z,\chi)\cdot \prod_p \left(1-\frac{\chi(p^2)}{p^{2+2z}}\right) \\ \qquad\qquad = L(1+z,\chi) \sum_{d=1}^{\infty} \frac{\mu(d)\,\chi(d^2)}{d^{2+2z}}. \end{cases}$$

When χ is not a principal character (i.e. $-k$ is not a square) $L(1+z,\chi)$ is regular for $\mathscr{R}(z)>-1$;

$$\sum_{d=1}^{\infty} \frac{\mu(d)\,\chi(d^2)}{d^{2+2z}}$$

defines a regular analytic function for $\mathscr{R}(z)>-{}^1/_2$. From *Eq.'s* (46), (47) and the principle of analytic continuation, $F_z(n,m)$ is regular for $\mathscr{R}(z)>-{}^1/_2$.

When $-k$ is a square, $(k\neq 0)$, χ is a principal character and $L(1+z,\chi)$ can be decomposed as:

$$(48)\qquad L(1+z,\chi) = \prod_{p\,|\,k} (1-p^{-1-z})\,(1-2^{-1-z})\,\zeta(1+z).$$

where ζ denotes the Riemann-Zeta function. (See LANDAU [1, p. 423]). Given the well-known properties of $\zeta(1+z)$, the equations (46) to (48) give the analytic continuation of $F_z(n,m)$ into the region $\mathscr{R}(z)>-{}^1/_2$ and show that it is regular there except for (at most) a simple pole at $z=0$. That it does have a pole follows if the other factors of $F_z(n,m)$ have no zeros there. The only possibility is $\prod_{p\,|\,k} \chi_{p,z}(k)$: but when $-k$ is a square, $k\neq 0$, (7e) gives, if $p^{2b}\|k$, (b a positive integer).

$$\begin{aligned} \chi_{p,z}(k)\Big|_{z=0} &= \chi_p(k) = \left(1+\frac{p-1}{p^2}+\frac{p-1}{p^3}+\ldots+\frac{p-1}{p^{b+1}}+\frac{1}{p^{b+1}}\right) \\ &= 1+1/p. \end{aligned}$$

We have now to compute $F_z(n,m)$ when $k=0$. By (7e)

$$(49)\qquad \begin{cases} \chi_{p,z}(0) = 1+\frac{p-1}{p^{2+2z}}+\frac{p-1}{p^{3+4z}}+\ldots \\ \qquad\quad = \frac{1-p^{-2-2z}}{1-p^{-1-2z}} \qquad (\sigma>-{}^1/_2). \end{cases}$$

If $\sigma>0$, we have $F_z(n,m)$ defined by an absolutely convergent product:

$$(50)\qquad \begin{cases} F_z(n,m) = \prod_p \chi_{p,z}(0) = \prod_p \frac{1-p^{-2-2z}}{1-p^{-1-2z}} \\ \qquad\qquad = \frac{1-2^{-1-2z}}{1-2^{-2-2z}}\,\frac{\zeta(1+2z)}{\zeta(2+2z)}. \end{cases}$$

The last member of (50) is regular for $\mathscr{R}(z) > -1/2$ except for a simple pole at $z=0$ and this completes the proof of the lemma.

Lemma 5:

a) $\chi_{2,z}(n, m)$ is an analytic function of z, regular at least in the region $\mathscr{R}(z) > -1/2$ for all n, m.

b) For those values of n and m for which $k=0$ or $-k$ is a square, $\chi_{2,z}(n, m)$ has a zero at $z=0$.

Proof. First consider $k \neq 0$. Inspection of (7a) to (7d) reveals that the series

$$\sum_{\lambda=1}^{\infty} B_{2^\lambda}(n, m)\, 2^{-\lambda z}$$

has at most $\alpha+1$ terms, when $2^\alpha \| k$ (α being a non-negative integer). Each term is an entire function of z, thus also $\chi_{2,z}(n, m)$.

Since also

$$|B_{2^\lambda}(n, m)| < 4.2^{-\lambda/2},$$

it follows easily that

$$|\chi_{2,z}(n, m)| < C_5 2^\alpha, \qquad (\sigma \geqslant -1/2,\ k \neq 0). \tag{51}$$

When $k=0$, $\chi_{2,z}(n, m)$ is not necessarily a finite sum of terms. It depends on the residue classes mod 4 or mod 8 to which n and m belong. The condition:

$$m^2 = 4n$$

restricts the possible $n.\,m$ combinations to the following: (1) $2\|m$, thus $2 \nmid n$ or (2) $4|m$ and $4|n$. If λ represents a positive integer, then from (7a) to (7d) we see that $B_{2^\lambda}(n, m)$ is zero for all allowable pairs n, m except as listed below in (52):

$$B_{2^\lambda}(n, m) = \begin{cases} (-1)^n & \text{if } \lambda = 1 \\ -2/2^{\lambda/2} & \text{if } 2|\lambda,\ \lambda \geqslant 2 \\ & \text{and } 4|m,\ 4|n. \end{cases} \tag{52}$$

The corresponding expressions for $\chi_{2,z}(n, m)$ emerge as:

$$\chi_{2,z}(n, m) = \begin{cases} 0 \quad \text{if} \quad 2\|m,\ 2\nmid n. \\ \dfrac{(2^z-1)(2^{1+z}+1)}{2^{2z}(1-2^{-1-2z})} \ \text{if } 4|m,\ 4|n, \end{cases} \qquad (\sigma \geqslant -1/2). \tag{53}$$

This completes the proof of a) and, in fact, part of b): for (53) shows that

$$\chi_{2,0}(n, m) = 0\ .$$

To continue with b): when $-k$ is a square ($k<0$), the possible combinations of n, m are again restricted. $B_{2^\lambda}(n, m)$ can be evaluated from (7a) to (7d) for all permissible combinations (see Eq. (54) for the possibilities) and $\chi_{2,z}(n, m)$ calculated thereby. The expressions for $\chi_{2,z}(n, m)$ after simplification, are:

$$\chi_{2,z}(n, m) = \begin{cases} 0 \quad \text{if } 2 \nmid m,\ 2 \mid n \text{ or if} \\ \quad 2 \| m,\ 2 \nmid n \text{ or if } 4 \mid m,\ 2 \nmid n, & \text{(a)} \\ 2^{-2z}(2^{1+2z}+1)\,(2^z-1) \text{ if } 2 \| m,\ 8 \mid n, & \text{(b)} \\ \dfrac{(2^{a+2az}-1)\,(2^{1+z}+1)\,(2^z-1)}{(2^{1+2z}-1)\,2^{a-1+2az}} \text{ if } 4 \mid m,\ 4 \mid n & \text{(c)}, \end{cases} \tag{54}$$

where, in (c), a is an integer such that $2^{2a} \| k$; clearly $a \geqslant 2$.

As before the factor $2^z - 1$ ensures that:

$$\chi_{2,0}(n, m) = 0,$$

when $k \neq 0$, and $-k$ is a square.

Lemma 6. a) The function $G_z(n, m)$ has an analytic continuation into the half-plane $\mathscr{R}(z) > -1/2$ which is regular there.

b) When $-k$ is not a square,

$$G_z(n, m)|_{z=0} = \sum_{q=1}^{\infty} B_q(n, m). \tag{55}$$

Proof. Part a) is an immediate consequence of equation (41) and lemmas 4 and 5.

To prove b) we observe that when χ is non-principal (mod $|4k|$)

$$L(1, \chi) = \sum_{j=1}^{\infty} \frac{\chi(j)}{j} = \sum_{j=1}^{\infty} \left(\frac{-4k}{j}\right) j^{-1},$$

where the series on the right is convergent (LANDAU [2, satz. 141]). Thus the equation (obtained from (47) by letting $z = 0$):

$$\sum_{q=1}^{\infty} \chi(q)\, \mu^2(q)/q = \sum_{d=1}^{\infty} \frac{\mu(d)\, \chi(d^2)}{d^2} \sum_{j=1}^{\infty} \frac{\chi(j)}{j}$$

has more than formal significance: for the left member of the equation is the Dirichlet product of the absolutely convergent and the convergent series occurring on the right hand side.

Thus from (46) and (41):

$$G_z(n, m)|_{z=0} = \chi_2(n, m) \prod_{p \mid k} \chi_p(k) \sum_{\substack{q=1 \\ (q,4k)=1}}^{\infty} B_q(n, m) = \sum_{q=1}^{\infty} B_q(n, m).$$

Lemma 7. In the set S,

$$|G_z(n, m)| < C_6\, k^2. \tag{56}$$

Proof. When $k = 0$, $\chi_{2,z}(n, m)$ and $F_z(n, m)$ are given by (53) and (50) respectively. The product is regular in S and independent of k, n or m: thus

$$|G_z(n, m)| < C_7 \qquad (k = 0). \tag{57}$$

When $-k$ is not a square, χ is non-principal and thus [1]):

$$(58) \qquad |L(1+z, \chi)| < C_8 |k|.$$

We next observe that

$$\sum_{d=1}^{\infty} \frac{\mu(d)\, \chi(d^2)}{d^{2+2z}}$$

defines a function independent of k and regular, thus bounded, in S. From equations (41), (46) and (47) and inequalities (44) and (51) (valid when $k \neq 0$) we have:

$$(59) \qquad |G_z(n, m)| < C_9 k^2, \qquad (-k \text{ non-square}).$$

When $k<0$ and $-k$ is a square $L(1+z, \chi)$ is given by (48) and $\chi_{2,z}(n, m)$ by (54). Thus the product $\chi_{2,z}(n, m)\cdot\zeta(1+z)$ is bounded in S. Since further

$$(60) \qquad \prod_{p \mid k} |(1-p^{-1-z})| < \prod_{p \mid k} p, \qquad (\sigma \geqslant -{}^1/_2)$$

and

$$|(2^{a+2az}-1)\; 2^{1-a-2az}| < 4 \qquad (\sigma \geqslant -{}^1/_2),$$

it easily follows that

$$(61) \qquad |G_z(n, m)| < C_{10} k^2, \qquad (k<0,\; -k \text{ a square})$$

This completes the proof of lemma 7.

Theorem 1. The functions $\psi_{m,z}(\tau), \psi_z(v|\tau)$ and $\theta_z(v|\tau)$ can be continued analytically into the half-plane $\mathscr{R}(z) > -{}^1/_2$. All three functions are regular in this region; and at $z=0$, they are respectively identical with $\psi_m(\tau)$, $\psi(v|\tau)$ and $\theta(v|\tau)$ as defined in (24), (17) and (12).

Proof. Recall from (39) and (40) that:

$$(62) \qquad \psi_{m,z}(\tau) = \frac{s^{-{}^1/_2}\, e\left(\frac{s-1}{8}\right)}{2^{(s-1)/2+z}} \sum_{k \equiv -m^2(s)} I(k, \tau, z)\, G_z(n, m) \qquad (\mathscr{R}(z) > {}^1/_2).$$

By lemmas 2 and 6, the terms of the series in (62) are functions regular in z for $\mathscr{R}(z) > -{}^1/_2$. From lemmas 2 and 7 it follows that the series:

$$C_{11} + \sum_{k \neq 0} C_{12}\, k^2 \cdot e^{-|k| C_2}$$

majorizes the series in (62) in set S. Thus the latter series defines an analytic function of z regular in S. Since S is an arbitrary set in the region $\mathscr{R}(z) > -{}^1/_2$, containing the neighbourhood

$$\{z : -{}^1/_2 < \sigma < 1,\; |t| < 1\},$$

$\psi_{m,z}(\tau)$ has an analytic extension regular for $\mathscr{R}(z) > -{}^1/_2$.

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1) See LANDAU [2, satz 235].

The series in (62) is uniformly convergent in S; thus on using lemmas 3 and 6 we have (by (24)):

$$(63)\quad \begin{cases} \psi_{m,0}(\tau) = \dfrac{\pi^{(s-1)/2}}{s^{(s-2)/2}\cdot\Gamma\left(\dfrac{s-1}{2}\right)} \displaystyle\sum_{\substack{k>0\\ k\equiv -m^2(s)}} e^{\pi i\tau k}\cdot k^{(s-3)/2}\sum_{q=1}^{\infty} B_q((k+m^2)/s,\, m) \\ \qquad\quad = \psi_m(\tau). \end{cases}$$

From (32) and (33) we get at once:

$$(64)\qquad \psi_0(v|\tau) = \psi_z(v|\tau)|_{z=0} = \psi(v|\tau)$$

$$(65)\qquad \theta_0(v|\tau) = \theta_z(v|\tau)|_{z=0} = \theta(v|\tau).$$

(To be continued)